THE PHILOSOPHICAL FOUNDATIONS OF MATHEMATICS.

I. HISTORICAL INTRODUCTION.

MATHEMATICS as commonly taught in our schools is based upon axioms. These axioms so called are a few simple formulas which the beginner must take on trust.

Axioms are defined to be self-evident propositions, and are claimed to be neither demonstrable nor in need of a demonstration. They are statements which are said to command the assent of every one who comprehends their meaning.

Euclid does not use the term "axiom." He starts with Definitions (ὅροι), which describe the meanings of point, line, surface, plane, angle, etc. He then proposes Postulates (αἰτήματα) in which he takes for granted that we can draw straight lines from any point to any other point, and that we can prolong any straight line in a straight direction. Finally, he adds what he calls Common Notions (κοιναὶ ἔννοιαι) which declare that things which are equal to the same thing are equal to one another; that if equals be added to equals, the wholes are equal, etc.

I need not mention here that the readings of the several manuscripts vary, and that some propositions (e. g., that all right angles are equal to one another) are now missing, now counted among the postulates, and now adduced as common notions.

In our modern editions of Euclid we find a statement concerning parallel lines added either to the Postulates or Common Notions. Originally it appeared in Proposition 29 where it is needed to prop up the argument that proves the equality of alternate angles on parallels, viz., that—

"those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles, do meet if continually produced."

This is exactly the point to be proved, for it is only one form of the conception of parallelism. So it was formulated axiomatically as follows:

"If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles."

This dogmatic statement (now commonly called "the axiom of parallels") was naturally transferred by the editors of Euclid to the introductory portion of the book where it now appears either as the fifth Postulate or the eleventh, twelfth, or thirteenth Common Notion. The latter is obviously the less appropriate place, for the idea of parallelism is assuredly not a Common Notion; it is not a rule of pure reason such as would be an essential condition of all thinking, reasoning, or logical argument. And if we do not give it a place of its own, it should either be classed among the postulates, or recast so as to become a pure definition.

It seems to me that no one can read the axiom of parallels as it stands in Euclid without receiving the impression that the statement was affixed by a later redactor. Even in Proposition 29, the original place of its insertion, it comes in as an afterthought; and if Euclid himself had considered the difficulty of the parallel axiom, so called, he would have placed it among the postulates in the first edition of his book, or formulated it as a definition.[1]

Though the axiom of parallels must be an interpolation, it is of classical origin, for it was known even to Proclus (410–485 A. D.), the oldest commentator of Euclid.

---

[1] For Professor Halsted's ingenious interpretation of the origin of the parallel theorem see *The Monist*, Vol. IV., No. 4, p. 487. He believes that Euclid anticipated metageometry, but it is not probable that the man who wrote the argument in Proposition 29 had the fifth Postulate before him. He would have referred to it or stated it at least approximately in the same words. But the argument in Proposition 29 differs considerably from the parallel axiom itself.

By the irony of fate, the theory of the parallel axiom, the authorship of which is extremely doubtful, has become more closely associated with Euclid's name than anything he has actually written, and when we now speak of Euclidean geometry we mean a system based upon this conception of parallelism.

The word axiom (ἀξίωμα) means "honor, reputation, high rank, authority," and is used by Aristotle, almost in the modern sense of the term, as "a self-evident highest principle," or "a truth so obvious as to be in no need of proof." It is connected with the verb ἀξιοῦν, "to deem worthy, to think fit, to maintain," and with ἄξιος, "worth" or "worthy."

The commentators of Euclid who did not understand the difference between Postulates and Common Notions, spoke of both as axioms, and even to-day the term Common Notion is mostly so translated.

We may state here at once that all the attempts made to derive the axiom of parallels from pure reason were necessarily futile, for no one can prove the absolute straightness of lines, or the evenness of space, by logical argument. Therefore these concepts, including the theory concerning parallels, cannot be derived from pure reason; they are not Common Notions and possess a character of their own. But the statement seemed thus to hang in the air, and there appeared the possibility of a geometry, and even of several geometries, in whose domains the parallel axiom would not hold good. This large field has been called metageometry, hypergeometry, or pangeometry, and may be regarded as due to a generalisation of the space-conception involving what might be called a metaphysics of mathematics.

Mathematics is a most conservative science. Its system is so rigid and all the details of geometrical demonstration are so complete, that the science was commonly regarded as a model of perfection. Thus the philosophy of mathematics remained undeveloped almost two thousand years. Not that there were not great mathematicians, giants of thought, men like the Bernouillis, Leibnitz and Newton, Euler, and others, worthy to be named in one breath with Archimedes, Pythagoras, and Euclid, but they ab-

stained from entering into philosophical speculations, and the very idea of a pangeometry remained foreign to them. They may privately have reflected on the subject, but they did not give utterance to their thoughts, at least they left no records of them to posterity.

It would be wrong, however, to assume that the mathematicians of former ages were not conscious of the difficulty. They always felt that there was a flaw in the Euclidean foundation of geometry, but they were satisfied to supply any need of basic principles in the shape of axioms, and it has become quite customary (I might almost say orthodox) to say that mathematics is based upon axioms. In fact, people enjoyed the idea that mathematics, the most lucid of all the sciences, was at bottom as mysterious as the most mystical dogmas of religious faith.

Metageometry has occupied a peculiar position among mathematicians as well as with the public at large. The mystic hailed the idea of *n*-dimensional spaces of curvature and other concepts of which we can form definite concepts but can have no concrete representations. They promised to justify all his speculations and give ample room for innumerable notions that otherwise would be doomed to irrationality. In a word, metageometry has always proved attractive to erratic minds. Among the professional mathematicians, however, those who were averse to philosophical speculation looked upon it with deep distrust, and therefore either avoided it altogether or rewarded its labors with bitter sarcasm. Prominent mathematicians did not dare to risk their reputation, and consequently many valuable thoughts must have remained unpublished. Even Gauss did not dare to speak out boldly, but communicated his thoughts to his most intimate friends under the seal of secrecy, not unlike a religious teacher who fears the odor of heresy. He did not mean to suppress his thoughts but he wanted to bring them before the public only in their mature shape. A letter to Taurinus concludes with the remark:

"Of a man who has proved himself a thinking mathematician, I fear not that he will misunderstand what I say, but under all circumstances you have to regard it merely as a private communication of which in no wise public use, or one that

might lead to it, is to be made. Perhaps I shall publish them myself in the future if I should gain more leisure than my circumstances at present permit."

C. F. Gauss.

Göttingen, 8. November, 1824.

The first attempt at dispensing with the problem of parallelisms was made by Nasir Eddin (1201–1274) whose work on Euclid was printed in Arabic in 1594 in Rome. His labors were taken up by John Wallis who in 1651 communicated Nasir Eddin's expositions of the fifth Postulate in a Latin translation to the mathematicians of the University of Oxford, and then propounded his own views in a lecture delivered on July 11, 1663. Nasir Eddin takes his stand upon the postulate that two straight lines which cut a third straight line, the one at right angles, the other at some other angle, will converge on the side where the angle is acute and diverge where it is obtuse. John Wallis, in his endeavor to prove this postulate, starts with the auxiliary theorem:

> If a limited straight line which lies upon an unlimited straight line be prolonged in a straight direction, its prolongation will fall upon the unlimited straight line.

There is no need of entering into the details of his proof. We may call his method the proposition of the straight line and will grant him that he proves the straightness of the straight line. In his further argument John Wallis shows the close connection of the problem of parallels with the notion of similitude. If we grant the one we must accept the other, and thus in Euclidean geometry the theorem holds good, for of any figure we can construct similar figures of any size.

Girolamo Saccheri, a learned Jesuit of the seventeenth century,[1] attacked the problem in a new way. He saw the close connection of parallelism with the right angle, and in his work on Eu-

[1] Born Sept. 5, 1667, in San Remo. He became a member of the Jesuit order March 24, 1685, and served as a teacher of grammar at the Jesuit College di Brera, in Milan, his mathematical colleague being Tommaso Ceva (a brother of the more famous Giovanni Ceva). Later on he became Professor of Philosophy and Polemic Theology at Turin and in 1697 at Pavia. He died in the College di Brera Oct. 25 1733.

clid[1] proposed three possibilities. If in the quadrilateral figure $ABCD$ the sides $AB = CD$ and the angles in A and B are right angles, the angles at C and D may be acute or obtuse. He undertakes to prove the absurdity of these assumptions so as to leave the only solution the sole possibility left, viz., that they must be right angles. But he finds difficulty in pointing out the contradiction to which these assumptions may lead and thus he opens a path on which Lobatchevsky (1793–1856) and Bolyai (1802–1860) followed, reaching a new view which makes three geometries possible, viz., the geometries of (1) the acute angle, (2) the obtuse angle, and (3) the right angle, the latter being the Euclidean geometry, in which the theorem of parallels holds.

While Saccheri seeks the difficulty of the problem in the notion of the right angle in the special case of a quadrangle, the German mathematician Lambert[2] starts from the notion of the sum of the angles of a triangle being equal to 180 degrees. There are three possibilities: the sum may be exactly equal to, more than, or less than 180 degrees. The first will make the triangle a figure in a plane, the second renders it spherical, and the third produces a geometry on the surface of an imaginary sphere. As to the last hypothesis Lambert said not without humor:[3]

"This result[4] possesses something attractive which easily suggests the wish that the third hypothesis might be true."

He then adds:[5]

"But I do not wish it in spite of these advantages, because there would be innumerable other inconveniencies. The trigonometrical tables would become in-

---

[1] *Euclides ab omni naevo vindicatus; sive conatus geometricus quo stabiliuntur prima ipsa universae geometriae principia.* Auctore Hieronymo Saccherio Societatis Jesu. Mediolani, 1773.

[2] Johann Heinrich Lambert was born August 26, 1728, in Mühlhausen, a city which at that time was a part of Switzerland. He died in 1777. His *Theory of the Parallel Lines*, written in 1766, was not published till 1786, nine years after his death, by Bernoulli and Hindenburg in the *Magazin für die reine und angewandte Mathematik*.

[3] P. 351, last line in the *Mag. für die reine und angewandte Math.*, 1786.

[4] That if in the square $ABCD$, the angle in G were less than 90 degrees (Lambert says 80 degrees), the side AB would be equal to AD.

[5] *Ibid.*, p. 352.

finitely more complicated and the similitude as well as proportionality of figures would cease altogether. No figure could be represented except in its own absolute size; and astronomy would be in a bad plight, etc."

Lobatchevsky's geometry is an elaboration of Lambert's third hypothesis and it has been called "imaginary geometry" because its trigonometric formulas are those of the spherical triangle if its sides are imaginary, or, as Wolfgang Bolyai has shown, if the radius of the sphere is assumed to be imaginary $=\sqrt{-1}$.

Lambert lays much stress on the proposition that each of the three hypotheses must be regarded as valid if only one case of it be found to be valid.

France has contributed least to the literature on the subject. Augustus de Morgan records the following story concerning the efforts of her greatest mathematician to solve the Euclidean problem. Lagrange, he says, composed at the close of his life a discourse on parallel lines. He began to read it in the Academy but suddenly stopped short and said: "Il faut que j'y songe encore." With these words he pocketed his papers and never recurred to the subject.

Legendre's treatment of the subject appears in the third edition of his elements of Euclid, but he omitted it from later editions as too difficult for beginners. Like Lambert he takes his stand upon the notion of the sum of the angles of a triangle, and like Wallis he relies upon the idea of similitude, saying that "the length of the units of measurement is indifferent for proving the theorems in question."[1]

A new epoch begins with Gauss, or rather with his ingenious disciple Riemann. While Gauss was rather timid about speaking openly on the subject, he did not wish his ideas to be lost to posterity. In a letter to Schumacher, dated May 17, 1831, he said:

"I have begun to jot down something of my own meditations, which are partly older than forty years, but which I have never written out, being obliged therefore to excogitate many things three or four times over. I do not wish them to pass away with me."

[1] *Mémoires de l'Académie des Sciences de l'Institut de France.* Vol. XII. 1833.

The notes to which Gauss here refers have not been found among his posthumous papers, and it therefore seems probable that they are lost and our knowledge of his thoughts remains limited to the comments that are scattered through his correspondence with mathematical friends.

Gauss wrote Bessel (1784–1846) January 27, 1829:

"I have also in my leisure hours frequently reflected upon another problem, now of nearly forty years' standing. I refer to the foundations of geometry. I do not know whether I have ever mentioned to you my views on this matter. My meditations here also have taken more definite shape, and my conviction that we cannot thoroughly demonstrate geometry *a priori* is, if possible, more strongly confirmed than ever. But it will take a long time for me to bring myself to the point of working out and making public my *very extensive* investigations on this subject, and possibly this will not be done during my life, inasmuch as I stand in dread of the clamors of the Bœotians, which would be certain to arise, if I should ever give *full* expression to my views. It is curious that *in addition to* the celebrated flaw in Euclid's Geometry, which mathematicians have hitherto endeavored in vain to patch and never will succeed, there is still another blotch in its fabric to which, so far as I know, attention has never yet been called and which it will by no means be easy, if at all possible, to remove. This is the definition of a plane as a surface in which a straight line joining *any two* points lies *wholly* in that plane. This definition contains *more* than is requisite to the determination of a surface, and tacitly involves a theorem which is in need of prior proof."

Bessel in his answer to Gauss makes a distinction between Euclidean geometry as practical and metageometry (the one that does not depend upon the theorem of parallel lines) as true geometry. He writes under the date of February 10, 1829:

"I should regard it as a great misfortune if you were to allow yourself to be deterred by the 'clamors of the Bœotians' from explaining your views of geometry. From what Lambert has said and Schweikardt orally communicated, it has become clear to me that our geometry is incomplete and stands in need of a correction which is hypothetical and which vanishes when the sum of the angles of a plane triangle is equal to 180°. This would be the *true* geometry and the Euclidean the *practical*, at least for figures on the earth."

In another letter to Bessel, April 9, 1830, Gauss sums up his views as follows:

"The ease with which you have assimilated my notions of geometry has been a source of genuine delight to me, especially as so few possess a natural bent for

them. I am profoundly convinced that the theory of space occupies an entirely different position with regard to our knowledge *a priori* from that of the theory of numbers (*Grössenlehre*); that perfect conviction of the necessity and therefore the absolute truth which is characteristic of the latter is totally wanting to our knowledge of the former. We must confess in all humility that a number is *solely* a product of our mind. Space, on the other hand, possesses also a reality outside of our mind, the laws of which we cannot fully prescribe *a priori*."

Another letter of Gauss may be quoted here in full. It is a reply to Taurinus and contains an appreciation of his essay on the Parallel Lines. Gauss writes from Göttingen, Nov. 8, 1824:

"Your esteemed communication of October 30th, with the accompanying little essay, I have read with considerable pleasure, the more so as I usually find no trace whatever of real geometrical talent in the majority of the people who offer new contributions to the so-called theory of parallel lines.

"With regard to your effort, I have nothing (or not much) more to say, except that it is incomplete. Your presentation of the demonstration that the sum of the three angles of a plane triangle cannot be greater than 180°, does indeed leave something to be desired in point of geometrical precision. But this could be supplied, and there is no doubt that the impossibility in question admits of the most rigorous demonstration. But the case is quite different with the second part, viz., that the sum of the angles cannot be smaller than 180°; this is the real difficulty, the rock on which all endeavors are wrecked. I surmise that you have not employed yourself long with this subject. I have pondered it for more than thirty years, and I do not believe that anyone could have concerned himself more exhaustively with this second part than I, although I have not published anything on this subject. The assumption that the sum of the three angles is smaller than 180° leads to a new geometry entirely different from our Euclidean,—a geometry which is throughout consistent with itself, and which I have elaborated in a manner entirely satisfactory to myself, so that I can solve every problem in it with the exception of the determining of a constant, which is not *a priori* obtainable. The larger this constant is taken, the nearer we approach the Euclidean geometry, and an infinitely large value will make the two coincident. The propositions of this geometry appear partly paradoxical and absurd to the uninitiated, but on closer and calmer consideration it will be found that they contain in them absolutely nothing that is impossible. Thus, the three angles of a triangle, for example, can be made as small as we will, provided the sides can be taken large enough; whilst the area of a triangle, however great the sides may be taken, can never exceed a definite limit, nay, can never once reach it. All my endeavors to discover contradictions or inconsistencies in this non-Euclidean geometry have been in vain, and the only thing in it that conflicts with our reason is the fact that if it were true there would necessarily exist in space a linear magnitude quite *determinate in it-*

self, yet unknown to us. But I opine that, despite the empty word-wisdom of the metaphysicians, in reality we know little or nothing of the true nature of space, so much so that we are not at liberty to characterise as *absolutely impossible* things that strike us as unnatural. If the non-Euclidean geometry were the true geometry, and the constant in a certain ratio to such magnitudes as lie within the reach of our measurements on the earth and in the heavens, it could be determined *a posteriori*. I have, therefore, in jest frequently expressed the desire that the Euclidean geometry should not be the true geometry, because in that event we should have an absolute measure *a priori*."

Schweikart, a contemporary of Gauss, may incidentally be mentioned as having worked out a geometry that would be independent of the Euclidean axiom. He called it astral geometry.[1]

Gauss's ideas fell upon good soil in his disciple Riemann (1826–1866) whose Habilitation Lecture on "The Hypotheses which Constitute the Bases of Geometry" inaugurates a new epoch in the history of the philosophy of mathematics.

Riemann states the situation as follows. I quote from Clifford's almost too literal translation (first published in *Nature*, 1873):

"It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

"From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it."

Riemann arrives at a conclusion which is negative. He says:

"The propositions of geometry cannot be derived from general notions of magnitude, but the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience."

In the attempt at discovering the simplest matters of fact from which the measure-relations of space may be determined, Riemann declares that—

---

[1] *Die Theorie der Parallellinien, nebst dem Vorschlag ihrer Verbannung aus der Geometrie.* Leipzig und Jena. 1807.

"Like all matters of fact, they are not necessary, but only of empirical certainty; they are hypotheses."

Being a mathematician, Riemann is naturally bent on deductive reasoning, and in trying to find a foothold in the emptiness of pure abstraction he starts with general notions. He argues that position must be determined by measuring quantities, and this necessitates the assumption that length of lines is independent of position. Then he starts with the notion of manifoldness, which he undertakes to specialise. This specialisation, however, may be done in various ways. It may be continuous, as is geometrical space, or consist of discrete units, as do arithmetical numbers. We may construct manifoldnesses of one, two, three, or n dimensions, and the elements of which a system is constructed may be functions which undergo an infinitesimal displacement expressible by dx. Thus, spaces become possible in which the directest linear functions (analogues to the straight lines of Euclid) cease to be straight and suffer a continuous deflection which may be positive or negative, increasing or decreasing.

Riemann argues that the simplest case will be, if the differential line-element ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx in which the coefficients are continuous functions of the quantities x, viz., $ds = \sqrt{\Sigma dx^2}$, but it is one instance only of a whole class of possibilities. He says:

"Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form $\sqrt{\Sigma dx^2}$, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call *flat*."

The Euclidean plane is the best-known instance of flat space being a manifold of a zero curvature.

Flat space has also been called by the new-fangled word *homaloidal* (from ὁμαλός, level), which recommends itself as a term in distinction from the popular meaning of even.

In applying his determination of the general notion of a manifold to actual space, Riemann expresses its properties thus:

"In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*; the former belongs to the extent-relations, the latter to the measure-relations. That space as an unbounded threefold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is forever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite, provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, i. e., a surface which in a *flat* manifoldness of three dimensions would take the form of a sphere, and consequently be finite."

It is obvious from these quotations that Riemann is a disciple of Kant. He was inspired mainly by his teacher Gauss and by Herbart. But while he starts a transcendentalist, employing mainly the method of deductive reasoning, he arrives at results which would stamp him as an empiricist of the school of Mill. He concludes that the nature of real space, which is only one instance among many possibilities, must be determined *a posteriori*. The problem of tridimensionality and homaloidality are questions which must be decided by experience, and while upon the whole he seems inclined to grant that Euclidean geometry is the most practical for a solution of the coarsest investigations, he is inclined to believe that real space is non-Euclidean. Though the derivation from the Euclidean standard can only be slight, there is a possibility of determining it by exact measurement and observation.

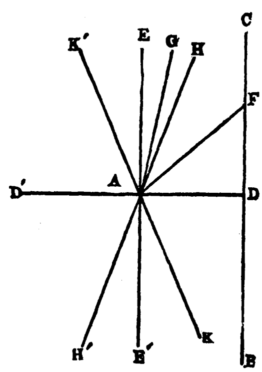
Riemann has succeeded in impressing his view upon metageometers down to the present day. They have built higher and introduced new ideas, yet the corner-stone of metageometry remained the same. It will therefore be found recommendable in a discussion of the problem to begin with a criticism of his Habilitation Lecture.

It is regrettable that Riemann was not allowed to work out his philosophy of mathematics. He died at the premature age of forty, but his work was continued by two other friends of Gauss,

Lobatchevsky and Bolyai, who actually contrived a geometry independent of the theorem of parallels.

It is perhaps no accident that the two independent and almost simultaneous inventors of a non-Euclidean geometry are original, not to say wayward, characters living on the outskirts of European civilisation, the one a Russian, the other a Magyar.

Nicolai Ivanovich Lobatchevsky[1] was born Oct. 22 (Nov. 2 of our calender), 1793, in the town of Makariev, about 40 miles above Nijni Novgorod on the Volga. His father was an architect; at his death in 1797 his widow and two small sons were left in poverty. At the gymnasium Lobatchevsky was noted for his wilfulness and disobedience, and he escaped expulsion only through the protection of his mathematical teacher, Professor Bartels, who even then recognised the extraordinary talents of the boy. Lobatchevsky graduated with distinction and became in his further career professor of mathematics and in 1827 Rector of the University of Kasan. Two books of his offered for official publication were rejected by the paternal government of Russia, and the manuscripts may be considered as lost for good. Of his several essays on the theories of parallel lines we mention only the one which made him famous throughout the whole mathematical world, *Geometrical Researches on the Theory of Parallels.*[2]



Lobatchevsky divides all lines, which in a plane go out from a point A with reference to a given straight line BC in the same plane, into two classes—cutting and not cutting. In progressing from the not-cutting lines, such as EA and GA, to the cutting lines, such as FA, we must come upon one HA that is the boundary between the two classes; and it is this which he calls the par-

[1] The name is spelled differently according to the different methods of transcribing Russian characters.

[2] For further details see Prof. G. B. Halsted's article "Lobachévski" in *The Open Court*, 1898, pp. 411 ff.

allel line. He designates the parallel angle on the perpendicular ($p = AD$, dropped from A upon BC) by Π. If $\Pi(p) < \frac{1}{2}\pi$ (viz., 90 degrees) we shall have on the other side of p another angle $DAK = \Pi(p)$ parallel to DB, so that on this assumption we must make a distinction of *sides in parallelism*, and we must allow two parallels, one on the one and one on the other side. If $\Pi(p) = \frac{1}{2}\pi$ we have only intersecting lines and one parallel; but if $\Pi(p) < \frac{1}{2}\pi$ we have two parallel lines as boundaries between the intersecting and non-intersecting lines.

We need not further develop Lobatchevsky's idea. Among other things, he proves that "if in any rectilinear triangle the sum of the three angles is equal to two right angles, so is this also the case for every other triangle," that is to say, each instance is a sample of the whole, and if one case is established, the nature of the whole system to which it belongs is determined.

The importance of Lobatchevsky's discovery consists in the fact that the assumption of a geometry from which the parallel axiom is rejected, does not lead to self-contradictions but to the conception of a general geometry of which the Euclidean is one possibility. This general geometry was later on most appropriately called by Lobatchevsky "Pangeometry."

John (or, as the Hungarians say, János) Bolyai imbibed the love of mathematics in his father's house. He was the son of Wolfgang (or Farkas) Bolyai, a fellow student of Gauss at Göttingen when the latter was nineteen years old and was professor of mathematics at Maros Vásárhely. Farkas wrote a two-volume book on the elements of mathematics[1] and in it he incidentally mentions his vain attempts at proving the axiom of parallels. His book was only partly completed when his son János wrote him of his discovery of a mathematics of pure space. He said:

"As soon as I have put it into order, I intend to write and if possible to publish a work on parallels. At this moment, it is not yet finished, but the way which I have followed promises me with certainty the attainment of my aim, if it is at all attainable. It is not yet attained, but I have discovered such magnificent things

[1] *Tentamen juventutem studiosam in elementa matheos* etc. *introducendi*, printed in Maros Vásárhely.

that I am myself astonished at the result. It would forever be a pity, if they were lost. When you see them, my father, you yourself will concede it. Now I cannot say more, only so much that *from nothing I have created another wholly new world.* All that I have hitherto sent you compares to it as a house of cards to a castle."[1]

János being convinced of the futility of proving Euclid's axiom, constructed a geometry of absolute space which would be independent of the axiom of parallels. And he succeeded. He called it the *Science Absolute of Space*,[2] an essay of twenty-four pages which Bolyai's father incorporated in the first volume of his *Tentamen* as an appendix.

Bolyai was a thorough Magyar. He was wont to dress in high boots, short wide Hungarian trousers, and a white jacket. He loved the violin and was a good shot. While serving as an officer in the Austrian army, János was known for his hot temper, which finally forced him to resign his commission as a captain, and we learn from Halsted that for some provocation he was challenged by thirteen cavalry officers at once. János calmly accepted and proposed to fight them all, one after the other, on condition that he be permitted after each duel to play a piece on his violin. We know not the nature of these duels nor the construction of the pistols, but the fact remains assured that he came out unhurt. As for the rest of the report that "he came out victor from the thirteen duels, leaving his thirteen adversaries on the square," we may be permitted to express a mild but deep-seated doubt.

János Bolyai starts with straight lines in the same plane, which may or may not cut each other. Now there are two possibilities: there may be a system in which straight lines can be drawn which do not cut one another, and another in which they all cut one another. The former, the Euclidean, he calls Σ, the latter S. "All theorems," he says, "which are not expressly asserted for Σ or for S are enunciated absolutely, that is, they are true whether Σ or S is

[1] See Halsted's introduction to the English translation of Bolyai's *Science Absolute of Space*, p. xxvii.

[2] *Appendix scientiam spatii absolute veram exhibens: a veritate aut falsitate axiomatis XI. Euclidei (a priori haud unquam decidenda) independentem; Adjecta ad casum falsitatis quadratura circuli geometrica.*

reality."[1] The system S can be established without axioms and is actualised in spherical trigonometry (*ibid. p.* 21). Now S can be changed to Σ, viz., plane geometry, by reducing the constant i to its limit (where the sect $y=0$) which is practically the same as the construction of a circle with $r=\infty$, thus changing its periphery into a straight line.

The labors of Lobatchevski and Bolyai are significant in so far as they prove beyond the shadow of a doubt that a construction of geometries other than Euclidean is possible and that it involves us in no absurdities or contradictions. This upset the traditional trust in Euclidean geometry as absolute truth, and it opened at the same time a vista of new problems, foremost among which was the question as to the mutual relation of these three different geometries.

It was Cayley who proposed an answer which was further elaborated by Felix Klein. These two ingenious mathematicians succeeded in deriving by projection all three systems from one common aboriginal form called by Klein *Grundgebild* or the Absolute. In addition to the three geometries hitherto known to mathematicians, Klein added a fourth one which he calls *elliptic.*[2]

Thus we may now regard all the different geometries as three species of one and the same genus and we have at least the satisfaction of knowing that there is *terra firma* at the bottom of our mathematics, though it lies deeper than was formerly supposed.

Prof. Simon Newcomb of Johns Hopkins University, although not familiar with Klein's essays, worked along the same line and arrived at similar results in his article on "Elementary Theorems Relating to the Geometry of a Space of Three Dimensions and of Uniform Positive Curvature in the Fourth Dimension.[3]

In the meantime the problem of geometry became interesting to outsiders also, for the theorem of parallel lines is a problem of

---

[1] See Halsted's translation, p. 14.

[2] "Ueber die sogenannte nicht-euklidische Geometrie" in *Math. Annalen*, 4, 6 (1871–1872). *Vorlesungen über nicht-euklidische Geometrie*, Göttingen. 1893.

[3] Crelle's *Journal für die reine und angewandte Mathematik*. 1877.

space. The best treatment of the subject came from the pen of the great naturalist Helmholtz whose two essays are so interesting because written in a most popular style.[1]

A collection of all the materials from Euclid to Gauss, compiled by Paul Stäckel and Friedrich Engel under the title *Die Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung zur Vorgeschichte der nicht-euklidischen Geometrie*, is perhaps the most useful and important publication in this line of thought, a book which has become indispensable to the student of metageometry and its history.

A store of information may be derived from Bertrand A. W. Russell's essay on the *Foundations of Geometry*. He divides the history of metageometry into three periods: The synthetic, consisting of suggestions made by Legendre and Gauss; the metrical, inaugurated by Riemann and characterised by Lobatchevsky and Bolyai; and the projective, represented by Cayley and Klein, who reduce metrical properties to projection and thus show that Euclidean and non-Euclidean systems may result from "the absolute."

Among modern writers no one has done more in the interest of metageometry than the indefatigable Dr. George Bruce Halsted, Professor of Mathematics in the University of Texas.[2] He has not only translated Bolyai and Lobatchevsky, but has also offered his own theories toward the solution of the problem.

Prof. B. J. Delbœuf and Prof. H. Poincaré have expressed their conceptions as to the nature of the bases of mathematics, in

---

[1] "Ueber die thatsächlichen Grundlagen der Geometrie," in *Wissenschaftl. Abh.*, 1866, Vol. II., p. 610 ff., and "Ueber die Thatsachen, die der Geometrie zum Grunde liegen," *ibid.*, 1868, p. 618 ff.

[2] From among his various publications we mention only his translations: *Geometrical Researches on the Theory of Parallels by Nicholaus Lobatchewsky.* Translated from the Original. And (2) *The Science Absolute of Space, Independent of the Truth or Falsity of Euclid's Axiom XI.* (*which can never be decided a priori*). *By John Bolyai.* Translated from the Latin, both published in Austin, Texas, the translator's present place of residence. Further, we refer the reader to Halsted's bibliography of the literature on hyperspace and non-Euclidean Geometry in the *American Journal of Mathematics*, Vol. I., pp. 261–276, 384, 385, and Vol. II., pp. 65–70.

articles contributed to *The Monist*.[1] The latter treats the subject from a purely mathematical standpoint, while Dr. Ernst Mach in his essay "On Physiological, as Distinguished from Geometrical, Space" (published in *The Monist* for April, 1901) attacks the problem in a very original manner and takes into consideration mainly the natural growth of space conception. His exposition might be called "the physics of geometry."

I cannot conclude this essay without paying a tribute to the memory of Hermann Grassmann of Stettin, a mathematician of first degree whose highly important results in this line of work have only of late found the recognition which they so fully deserve. I do not hesitate to say that Hermann Grassmann's *Lineare Ausdehnungslehre* is the best work on the philosophical foundation of mathematics from the standpoint of a mathematician.

Grassmann establishes first the idea of mathematics as the science of pure form. He shows that the mathematician starts from definitions. He then proceeds to show how the product of thought may originate either by the single act of creation, or by the double act of positing and combining. The former is the continuous form, or magnitude, in the narrower sense of the term, the latter the discrete form or the method of combination. He distinguishes between intensive and extensive magnitude and chooses as the best example of the latter the sect[2] or limited straight line. Hence the name of the new science, "theory of linear extension."

Grassmann constructs linear formations of which systems of one, two, three, and n degrees are possible. The Euclidean plane is a system of second degree, and space a system of third degree, He thus generalises the idea of mathematics, and having created a

---

[1] They are as follows: "Are the Dimensions of the Physical World Absolute?" by Prof. B. J. Delbœuf, *The Monist*, January, 1894; "On the Foundations of Geometry," by Prof. H. Poincaré, *The Monist*, October, 1898; also "Relations Between Experimental and Mathematical Physics," *The Monist*, July, 1902.

[2] Grassmann's term is *Strecke*, a word connected with the Anglo-Saxon "Stretch," being that portion of a line that stretches between two points. The translation "sect," than which there is no better, was suggested by Prof. G. B. Halsted.

science of pure form, points out that geometry is one of its applications which originates under definite conditions. Grassmann made the straight line the basis of his geometrical definitions. He defines the plane as the totality of parallels which cut a straight line and space as the totality of parallels which cut the plane. Here is the limit to geometrical construction, but abstract thought knows of no bounds. Having generalised our mathematical notions as systems of first, second, and third degree, we can continue in the numeral series and construct systems of four, five, and still higher degrees. Further, we can determine any plane by any three points, given in the figures x_1, x_2, x_3, not lying in a straight line. If the equation between these three figures be homogeneous, the totality of all points that correspond to it will be a system of second degree. If this homogeneous equation is of the first grade, this system of second degree will be simple, viz., of a straight line; but if the equation be of a higher grade, we shall have curves for which not all the laws of plane geometry hold good. The same considerations lead to a distinction between homaloidal space and non-Euclidean systems.[1]

Being professor at a German gymnasium and not at a university, Grassmann's book remained neglected and the newness of his methods prevented superficial readers from appreciating the sweeping significance of his propositions. Since there was no call whatever for the book, the publishers returned the whole edition to the paper mill, and the complimentary copies which the author had sent out to his friends are perhaps the sole portion that was saved from the general doom.

Grassmann, disappointed in his mathematical labors, had in the mean time turned to other studies and gained the honorary doctorate of the University of Tübingen in recognition of his meritorious work on the St. Petersburg Sanskrit Dictionary, when Victor Schlegel called attention to the similarity of Hamilton's theory of vectors to Grassmann's concept of *Strecke*, both being limited straight lines of definite direction. Suddenly a demand for Grass-

---

[1] See Grassmann's *Ausdehnungslehre* von 1844, Anhang I., pp. 273–274.

mann's book was created in the market; but alas! no copy could be had, and the publishers deemed it advisable to reprint the destroyed edition of 1844. The satisfaction of this late recognition was the last joy that brightened the eve of Grassmann's life. He wrote the introduction and an appendix to the second edition of his *Lineare Ausdehnungslehre*, but died while the forms of his book were on the press.

At the present day the literature on metageometrical subjects has grown to such an extent that we do not venture to enter into further details. We will only mention the appearance of Professor Schoute's work on more-dimensional geometry[1] which promises to be the elaboration of the pangeometrical ideal.

Having reviewed all the attempts made to establish non-Euclidean systems of geometry, it may not be out of place to state, that in spite of the well-deserved fame of the metageometricians from Wallis to Halsted, Euclid's claim to classicism remains unshaken. The metageometrical movement is not a revolution against Euclid's authority but an attempt at widening our mathematical horizon. Let us hear what Halsted, one of the boldest and keenest heroes among the champions of metageometry of the present day, has to say of Euclid. Halsted begins the Introduction to his English translation of *Bolyai's Science Absolute of Space* with a terse description of the history of his great book *The Elements of Geometry*, the rediscovery of which is not the least factor that initiated a new epoch in the development of Europe which may be called the era of inventions, of discoveries, and of the appreciation as well as growth of science. Halsted says:

"The immortal *Elements* of Euclid was already in dim antiquity a classic, regarded as absolutely perfect, valid without restriction.

"Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek, as the Parthenon. On this foundation pure science rose in Archimedes, in Apollonius, in Pappus; struggled in Theon, in Hypatia; declined in Proclus; fell into the long decadence of the Dark Ages.

---

[1] *Mehrdimensionale Geometrie* von Dr. P. H. Schoute, Professor der Math. an d. Reichs-Universität zu Groningen, Holland. Leipzig, Göschen's Verlag. So far only the first volume, which treats of linear space, has appeared.

"The book that monkish Europe could no longer understand was then taught in Arabic by Saracen and Moor in the Universities of Bagdad and Cordova.

"To bring the light, after weary, stupid centuries, to western Christendom, an Englishman, Adelhard of Bath, journeys, to learn Arabic, through Asia Minor, through Egypt, back to Spain. Disguised as a Mohammedan student, he got into Cordova about 1120, obtained a Moorish copy of Euclid's *Elements*, and made a translation from the Arabic into Latin.

"The first printed edition of Euclid, published in Venice in 1482, was a Latin version from the Arabic. The translation into Latin from the Greek, made by Zamberti from a MS. of Theon's revision, was first published at Venice in 1505.

"Twenty-eight years later appeared the *editio princeps* in Greek, published at Basle in 1533 by John Hervagius, edited by Simon Grynaeus. This was for a century and three-quarters the only printed Greek text of all the books, and from it the first English translation (1570) was made by 'Henricus Billingsley,' afterward Sir Henry Billingsley, Lord Mayor of London in 1591.

"And even to-day, 1895, in the vast system of examinations carried out by the British Government, by Oxford, and by Cambridge, no proof of a theorem in geometry will be accepted which infringes Euclid's sequence of propositions.

"Nor is the work unworthy of this extraordinary immortality.

"Says Clifford: 'This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state.

"'The encouragement; for it contained a body of knowledge that was really known and could be relied on.

"'The guide; for the aim of every student of every subject was to bring his knowledge of that subject into a form as perfect as that which geometry had attained.'"

Euclid's *Elements of Geometry* are not counted among the books of divine revelation, but truly they deserve to be held in religious veneration. There is a sanctity in mathematical truth which is not sufficiently appreciated, and certainly if truth, helpfulness, and directness and simplicity of presentation, give a title to rank as divinely inspired literature, Euclid's great work should be counted among the canonical books of mankind.

* * *

The author has purposely introduced what might be called a personal element in these expositions of a subject which is commonly regarded as dry and abstruse, and endeavored to give something of the lives of the men who have struggled and labored

in this line of thought and have sacrificed their time and energy on the altar of one of the noblest aspirations of man, the delineation of a philosophy of mathematics.

There is one more point which should be mentioned in this connection. It is the close analogy which mathematics bears to religion. We have first the rigid presentation of mathematical truth discovered (as it were) by instinct, by a prophetic divination, for practical purposes, in the shape of a dogma as based upon axioms, which is followed by a period of unrest, being the search for a philosophical basis, which finally leads to a higher standpoint from which, though it acknowledges the relativity of the primitive dogmatism, consists in a recognition of the eternal verities on which are based all our thinking, and being, and yearning.

Is there any need of warning our readers that this sketch of the history of metageometry is both brief and popular? We have purposely avoided the discussion of technical details, limiting our exposition to the most essential points and trying to show them in a light that will render them interesting to the non-mathematical reader also. The article is intended to serve as an introduction to a ventilation of the problem itself, and the author proposes to continue the subject in a discussion anent the foundation of mathematics—an article which he hopes to be able to submit to his readers in a subsequent number of *The Monist*. He has abstained from criticising the views of his predecessors because he wishes first to submit his own solution to the criticism of professional mathematicians.

EDITOR.